Journal of Engineering Physics and Thermophysics, Vol. 81, No. 6, 2008

HYDRO-GAS-DYNAMICS IN TECHNOLOGICAL PROCESSES

APPLICATION OF THE METHOD OF ASYMPTOTIC EXPANSIONS FOR ANALYZING FREE-CONVECTION JET FLOWS

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From the viewpoint of the laminar boundary layer theory, by the method of internal and external asymptotic expansions the self-similar problem on the development of a plane free-convective jet at a quadratic temperature dependence of the density has been solved. Analytical dependences of the main characteristics of the jet flow on the Prandtl number, convenient for engineering calculations, have been obtained.

Introduction. The progress in the field of computing techniques and the development of various algorithms for solving a wide range of mathematical problems have made it possible to employ numerical methods of investigations in various fields of natural science. In the hydromechanics and heat transfer problems, numerical simulation has become one of the main methods of analysis, and in recent years its role has been steadily increasing. However, in solving a problem by a purely numerical method, it is very difficult to form a complex idea of the physical process under investigation as well as to establish the dependences of the main characteristics on the diagnostic variables in a wide range of their variation. In view of this fact, of great importance are theoretical studies which help to elucidate the most important aspects of the investigated physical process and serve as a proving ground for testing and checking results obtained within the framework of different numerical schemes.

Free-convective jets [1] are of great interest from the technical point of view, since this phenomenon is common in the industry, in technological processes, and in nature. The knowledge of the laws of such flows has, e.g., a direct bearing on the formation of a gas-air screen for protecting the population against poisonous and toxic agents in the case of accidents at chemical enterprises and plants [2]. Moreover, the problem of jet momentum and heat transfer in the field of mass forces is of fundamental interest, since it incorporates the nonlinear interaction of the fields, the scalar, and the vector. Widening of the spheres of application of free-convective jet flows naturally calls for placing more stringent requirements upon the accuracy and reliability of calculations. Therefore, the development of methods of analysis and the construction of exact and approximate solutions in this field of knowledge are highly topical since they make it possible to obtain analytical dependences and perform, on their basis, qualitative and quantitative analysis of the process being investigated.

Below we present, from the viewpoint of the boundary layer model, the results of solving by the method of comparative asymptotic expansions the self-similar problem on the development of a plane free-convective jet.

Formulation of the Problem. Consider the vertical motion of a liquid caused by a stationary horizontal linear heat source of intensity Q_0. We assume all properties of the liquid to be constant except for the density $\rho = \rho_\infty (1 - \gamma(\Delta T)^2)$. Then the calculation of the flow field and the heat transfer in a laminar plane free-convective jet reduces to the integration of the system of equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\gamma(\Delta T)^2\,, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \quad u\frac{\partial \Delta T}{\partial x} + v\frac{\partial \Delta T}{\partial y} = \frac{\nu}{\Pr}\frac{\partial^2 \Delta T}{\partial y^2}\,. \tag{1}$$

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The boundary conditions take into account the presence of the jet flow symmetry and the absence of solid boundaries: in the environment, the axial velocity u goes to zero and the temperature T is equal to T_∞. Therefore,

$$y = 0: \quad v = \frac{\partial u}{\partial y} = \frac{\partial \Delta T}{\partial y} = 0 ; \quad y \to \infty : \quad u \to 0 , \quad \Delta T \to 0 . \tag{2}$$

The energy conservation law requires that at any $x > 0$ the energy transferred by convection be equal to the energy Q_0 released by the heat source:

$$Q_0 = \rho C_p \int_{-\infty}^{+\infty} u \Delta T dy = \text{const} . \tag{3}$$

Let us introduce the self-similar variables

$$\psi = \left(\frac{g\gamma\nu^2 Q_0^2}{\rho^2 C_p^2}\right)^{1/6} f x^{1/2} , \quad \eta = \left(\frac{g\gamma Q_0^2}{\rho^2 C_p^2 \nu^4}\right)^{1/6} x^{-1/2} y , \quad u = \left(\frac{g\gamma Q_0^2}{\rho^2 C_p^2 \nu}\right)^{1/3} f', \quad \Delta T = \left(\frac{Q_0^4}{\rho^4 C_p^4 g\gamma\nu^2}\right)^{1/6} h x^{-1/2} . \tag{4}$$

Applying transforms of the form (4) to Eqs. (1)–(3), we obtain a system of differential equations (a dash denotes derivative with respect to η)

$$f''' + \frac{1}{2} f f'' + h^2 = 0 , \quad \frac{1}{\text{Pr}} h'' + \frac{1}{2} f h' + \frac{1}{2} f' h = 0 , \quad f(0) = f''(0) = h'(0) = 0 ,$$
$$f'(\infty) = h(\infty) = 0 , \quad \int_0^\infty f' h d\eta = \frac{1}{2} . \tag{5}$$

Problem (5) describes the self-similar flow conditions of the liquid created at a certain distance from the source, i.e., expressions (4) represent asymptotic laws of the development of the investigated physical process. However, the introduction into the consideration of an additional parameter ("polar" distance of the jet) permits using formulas (4) also for calculating the velocity and temperature fields in the region of the nonself-similar development of the flow [1]. Thus, if the functions $f(\eta)$ an $h(\eta)$ are known, then we can describe the momentum and heat transfer processes in a free-convective jet, as well as forecast the laws of their change depending on various parameters. To solve system (5), let us make use of the method of joining asymptotic expansions. Note that the given approach based on the representation of the solution of boundary-value problems formulated in terms of equations in the form of asymptotic series has proved efficient in the theory of free-convective heat transfer [3], as well as being competitive compared to other analytic methods [4–8].

Solution of the Problem. Consider the case of large Prandtl numbers. At Pr >> 1 the boundary layer of the jet can be broken down into two regions: one region of thickness $\delta_T \sim \text{Pr}^{-7/12}$, in which the quantity ΔT tends to zero, and the other region of thickness $\delta_{\text{outer}} \sim \text{Pr}^{1/12}$, in which the velocity $u \to 0$. The above regions are called the outer and inner layers and will be considered separately from each other. To each region there corresponds its own mathematical model in the form of a boundary-value problem for different types of nonlinear differential equations. Let us introduce he following similarity conversions:

inner layer

$$f = \text{Pr}^{-5/12} F(\xi) , \quad \xi = \text{Pr}^{7/12} \eta , \quad h = \text{Pr}^{5/12} H(\xi) ; \tag{6}$$

outer layer

$$f = \text{Pr}^{1/12} G(z) , \quad z = \text{Pr}^{1/12} \eta , \quad h = 0 .$$

The functions $F(\xi)$, $H(\xi)$, and $G(z)$ satisfy the equations

$$F''' + \frac{1}{2\mathrm{Pr}} FF'' + \frac{1}{\mathrm{Pr}^{1/2}} H^2 = 0\,, \quad H' + \frac{1}{2} FH = 0\,, \quad \int_0^\infty F'Hd\xi = \frac{1}{2}\,, \quad G''' + \frac{1}{2} GG'' = 0\,, \tag{7}$$

where a dash denotes differentiation with respect to the corresponding variable ξ and z. Further we will represent $F(\xi)$ and $H(\xi)$ in the form of expansions. In so doing, according to (7), we will use the quantity $\varepsilon = \mathrm{Pr}^{-1/2}$ as the series expansion parameter

$$F(\xi) = F_0(\xi) + \varepsilon F_1(\xi) + \varepsilon^2 F_2(\xi) + \ldots\,, \quad H(\xi) = H_0(\xi) + \varepsilon H_1(\xi) + \varepsilon^2 H_2(\xi) + \ldots\,. \tag{8}$$

Series (8) are called internal asymptotic expansions. Let us substitute (8) into (7) and equate the coefficients at equal powers ε. We will obtain a recurrent system of equations for F_i and H_i:

$$F_0''' = 0\,, \quad F_1''' + H_0^2 = 0\,, \quad F_i''' + 2H_0H_{i-1} = -\frac{1}{2}\sum_{j=1}^{i-1} F_{j-1}F_{i-j-1}'' - \sum_{j=1}^{i-2} H_jH_{i-j-1}\,; \quad H_0' + \frac{1}{2} F_0H_0 = 0\,,$$

$$H_i' + \frac{1}{2} F_0H_i = -\frac{1}{2}\sum_{j=1}^{i} F_jH_{i-j}\,; \quad \int_0^\infty F_0'H_0d\xi = \frac{1}{2}\,, \quad \int_0^\infty \left(F_0'H_i + \sum_{j=1}^{i} F_j'H_{i-j}\right)d\xi = 0\,. \tag{9}$$

Hereinafter it is assumed that if the upper limit of summation is smaller than the lower one or if in one of the functions the index is negative, then such sums and functions should be assumed equal to zero.

We will also define the function $G(z)$ in the form of a series by the parameter ε:

$$G(z) = G_0(z) + \varepsilon G_1(z) + \varepsilon^2 G_2(z) + \ldots\,. \tag{10}$$

Then we will arrive at the recurrent system of equations for G_i:

$$G_0''' + \frac{1}{2} G_0G_0'' = 0\,, \quad G_1''' + \frac{1}{2} G_0G_1'' + \frac{1}{2} G_0''G_1 = 0\,, \quad G_i''' + \frac{1}{2} G_0G_i'' + \frac{1}{2} G_0''G_i = -\frac{1}{2}\sum_{j=1}^{i-1} G_jG_{i-j}''\,. \tag{11}$$

From (9) and (11) it is seen that the internal and external expansions are only related through the boundary conditions. Note, however, that relations (5) do not entail conclusively the boundary conditions of the auxiliary problems for F_i, H_i, and G_i (the conditions at $\eta \to \infty$ for F and at $\eta = 0$ for G are redundant). Consequently, the search for a solution of the problem in the form (8) and (10) requires the introduction of additional relations of "sewing" asymptotic series. In accordance with the consistency conditions, the following equalities should hold:

$$\lim_{\xi\to\infty}\left(F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \ldots\right) = \lim_{z\to 0}\left(\varepsilon G_0 + \varepsilon^2 G_1 + \varepsilon^3 G_2 + \ldots\right), \quad \lim_{\xi\to\infty}\left(H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \ldots\right) = 0\,. \tag{12}$$

As a result, restricting ourselves to three approximations, we obtain:

inner problem

$$F_0''' = 0\,, \quad H_0' + \frac{1}{2} F_0H_0 = 0\,, \quad \int_0^\infty F_0'H_0d\xi = \frac{1}{2}\,; \quad F_0(0) = F_0''(0) = F_0''(\infty) = H_0(\infty) = 0\,;$$

$$F_1''' + H_0^2 = 0\,, \quad H_1' + \frac{1}{2} F_0H_1 = -\frac{1}{2} F_1H_0\,, \quad \int_0^\infty \left(F_0'H_1 + F_1'H_0\right)d\xi = 0\,,$$

$$F_1(0) = F_1''(0) = H_1(\infty) = 0\,, \quad F_1''(\infty) = G_0''(0)\,; \quad F_2''' + 2H_0H_1 = -\frac{1}{2} F_0F_0''\,, \tag{13}$$

$$H_2' + \frac{1}{2} F_0 H_2 = -\frac{1}{2} F_1 H_1 - \frac{1}{2} F_2 H_0 \,, \quad \int_0^\infty \left(F_0' H_2 + F_1' H_1 + F_2' H_0 \right) d\xi = 0 \,,$$

$$F_2 (0) = F_2''(0) = H_2 (\infty) = 0 \,, \quad F_2''(\infty) = G_1''(0) \,;$$

outer problem

$$G_0''' + \frac{1}{2} G_0 G_0'' = 0 \,, \quad G_0 (0) = G_0' (\infty) = 0 \,, \quad G_0' (0) = F_0' (\infty) \,; \quad G_1''' + \frac{1}{2} G_0 G_1'' + \frac{1}{2} G_0'' G_1 = 0 \,,$$

$$G_1 (0) = \lim_{\xi \to \infty} \left(F_0 - \xi G_0' (0) \right), \quad G_1' (0) = \lim_{\xi \to \infty} \left(F_1' - \xi G_0''(0) \right), \quad G_1' (\infty) = 0 \,; \tag{14}$$

$$G_2''' + \frac{1}{2} G_0 G_2'' + \frac{1}{2} G_0'' G_2 = -\frac{1}{2} G_1 G_1'', \quad G_2 (0) = \lim_{\xi \to \infty} \left(F_1 - \xi G_1' (0) - \frac{\xi^2}{2} G_0''(0) \right),$$

$$G_2' (0) = \lim_{\xi \to \infty} \left(F_2' - \xi G_1''(0) \right), \quad G_2' (\infty) = 0 \,.$$

Analysis of (13) shows that the functions F_0 and H_0 depend on neither the higher approximations nor the solution of system (14). The zeroth-order approximation therewith is found by a simple integration. Then the boundary conditions for the first equation of system (14) are defined completely. Having solved this equation and having constructed the function G_0, we find the boundary conditions of the relation for the next approximation of the inner problem, and so on.

Consequently, the solution of the boundary-value problem (5) by the method of joining asymptotic expansions reduces to a step-by-step integration of the equations of the inner (13) and outer (14) problems, which permits attaining a relatively simple solution. Further on, such an approach makes it possible to obtain a solution of the problem up to any power ε. And, finally, the specificity of Eqs. (13) and (14), unlike (15), is that they do not contain the parameter Pr and, therefore, the functions $F_i(\xi)$, $H_i(\xi)$, and $G_i(z)$ will not depend on Pr. In other words, we find the correlation dependences of the sought characteristics on the Prandtl number. These dependences permit obtaining quantitative and qualitative information and elucidating the main features of the investigated physical process, which is of certain importance from the theoretical and applied (technological) viewpoints.

The zeroth-order approximation (8) is written in quadratures

$$F_0 = C_2 \xi \,, \quad H_0 = \frac{1}{\sqrt{4\pi C_2}} \exp \left(-\frac{C_2}{4} \xi^2 \right). \tag{15}$$

Knowing H_0, we find the function F_i:

$$F_1 = -\frac{1}{8\pi C_2} \xi^2 \int_0^\xi \exp \left(-\frac{C_2}{2} \xi^2 \right) d\xi - \frac{1}{8\pi C2} \xi \exp \left(-\frac{C_2}{2} \xi^2 \right) - \frac{1}{8\pi C_2^2} \int_0^\xi \exp \left(-\frac{C_2}{2} \xi^2 \right) d\xi + C_6 \xi \,. \tag{16}$$

The coefficients C_2 and C_6 have not yet been determined. They are calculated further in the process of solution. Using formulas (15) and (16), we obtain the following expression for H_1:

$$H_1 = -\frac{1}{4\sqrt{\pi C_2}} \exp \left(-\frac{C_2}{4} \xi^2 \right) \left(-\frac{1}{24\pi C_2} \xi^3 \int_0^\xi \exp \left(-\frac{C_2}{2} \xi^2 \right) d\xi \;\; \frac{1}{24\pi C_2^2} \xi^2 \exp \left(-\frac{C_2}{2} \xi^2 \right) \right.$$

$$\left. - \frac{1}{8\pi C_2^2} \xi \int_0^\xi \exp \left(-\frac{C_2}{2} \xi^2 \right) d\xi - \frac{1}{12\pi C_2^3} \exp \left(-\frac{C_2}{2} \xi^2 \right) + C_6 \frac{\xi^2}{2} + \frac{1}{C_2} \left(C_6 - \frac{\sqrt{3}}{4\pi C_2^2} \right) \right). \tag{17}$$

For further analysis, it is necessary to construct the functions G_0 and G_1. The nonlinearity of the system of equations (14) and the asymptotic character of the boundary conditions considerably complicate the integration of the outer problem. Therefore, the necessity of using numerical methods arises. Notice that the solution of the system of equations of zeroth- and first-order equations (14) is only possible at certain relations between the coefficients C_2 and C_6. This means that to select the sought solution one has to find that one which satisfies the condition on infinity. The latter makes it possible to reduce the boundary-value problems (14) to Cauchy problems with unknown parameters and use the shooting method. However, numerical integration of such a kind of equations faces certain difficulties connected with the strong dependence of the sought functions on the deficient boundary conditions at $z = 0$, since convergent solutions exist in a very narrow range of arbitrarily given quantities. Such behavior of equations is known in computational mathematics as a "rigid problem." Therefore, in the process of numerical counting it is desirable to have methods that permit overcoming the above difficulties. This can be done, for example, by finding preliminarily the unknown input parameters. The idea of the approach is the representation of functions $G_0(G_1)$ in the form of n terms, with the solution corresponding to the case where $G_0''' = 0$ is taken as the first approximation. Then three (four) terms of the series are constructed, and with their use the formulas for the estimating values of the quantities C_2 and C_6 are found. The last procedure corresponds to the summation of some infinite numerical sequence entering into the basic series. Note that the given method was put to the test in solving applied problems of jet hydrodynamics and proved to be reliable and efficient [1, 9, 10]. The proposed approach makes it possible to considerably reduce the computer time and preserve the main features of the problem and the advantages of the employed method of analysis. The calculations performed by the described algorithm enabled us to obtain the coupling equation and the value of the quantity C_2:

$$C_6 = \frac{2\sqrt{2} + 3\sqrt{3}}{36\pi C_2^2}, \quad C_2 = \left(\frac{311}{656}\right)^{2/3}. \tag{18}$$

The results obtained are in good agreement with the numerical integration data:

$$G_0''(0) = -0.2103746, \quad G_1''(0) = -0.0996188, \quad G_0'(0) = 0.6080012, \quad G_1'(0) = 0.1919380,$$

$$G_0(\infty) = 1.2601647, \quad G_1(\infty) = 0.1989087.$$

The quantities $G_0'(0) = G_2$ and $G_2'(0) = G_6$ calculated with the use of the Runge–Kutta–Merson scheme and relations (18) practically coincide: the maximum absolute error is $\sim 1 \cdot 10^7$. This points to the fact that the possibilities of analytical approaches to the investigation of convective heat transfer problems are still far from exhausted despite the fact that computer science is advancing steadily.

Using the constructed solutions in the inner region, determine $f'(0)$ and $h(0)$:

$$f'(0) = \Pr^{1/6}\left\{C_2 + \frac{2\sqrt{2} + 3\sqrt{3} - 9}{36\pi C_2^2}\Pr^{-1/2} + \ldots\right\} = \Pr^{1/6}\left\{0.6080011 - 0.0233309\Pr^{-1/2} + \ldots\right\},$$

$$h(0) = \Pr^{5/12}\left\{\frac{1}{\sqrt{4\pi C_2}} + \frac{6\sqrt{3} - 2\sqrt{2} + 3}{144\pi\sqrt{\pi}\, C_2^{7/2}}\Pr^{-1/2} + \ldots\right\} = \Pr^{5/12}\left\{0.3617786 + 0.0751745\Pr^{-1/2} + \ldots\right\}. \tag{19}$$

Because the asymptotic behavior of the zeroth and the first approximations is different, it will be assumed that expressions (19) obtained under the condition that Pr >> 1 can also be used in calculations at moderate values of Pr. To estimate the exactness of formula (19), we compared the values of $f'(0)$ and $h(0)$ to the data of numerical integration of the system of equations (5) [11]. If we take the 3% difference as a criterion, then we can say that agreement begins at Pr = 5. Let us now analyze the dependence of $f(\infty)$ on the Pr number. By virtue of the solutions in the outer region, the following representation of the function holds:

$$f(\infty) = \Pr^{1/12}\left\{1.2601647 + 0.1989087\Pr^{-1/2} + \ldots\right\}. \tag{20}$$

The error of the approximate formula (20) at Pr = 5 is 2.3%. From (20) it follows that the dependence of the mass rate of flow in the jet

$$\frac{m}{\mu} = \left(\frac{g\gamma Q_0^2}{\rho^2 C_p^2 \nu^4}\right)^{1/6} 2f(\infty)\, x^{1/2} \tag{21}$$

is different at different Pr numbers. Indeed, according to (11), with decreasing Pr number the value of $f(\infty)$ decreases to a certain minimum value and then its increase is observed. We can also see how the Pr number influences the temperature and velocity profiles. From (19) it follows that an increase in the Pr number is accompanied by an increase in u_m and ΔT_m as a consequence of the strengthening of the influence of buoyancy forces on the jet flow. The rate of rise of ΔT_m turns out to be somewhat higher than that of u_m, which is due to their asymptotic laws: $u_m \sim \text{Pr}^{1/6}$, $\Delta T_m \sim \text{Pr}^{5/12}$. Analyzing thereby formulas (19), it may be stated that the maximum velocity reaches its asymptotic dependence faster than the maximum temperature.

Conclusions. The results presented in this paper point to the high efficiency of the method of joining asymptotic expansions for solving problems of free-convective jet flows. The very construction of two (three) approximations, which is equivalent to keepign in (8) and (10) the first two (three) terms of the series, enables one to develop a simple and practically convenient mathematical apparatus for investigating the features and mechanisms of the process being studied. This fact may stimulate special calculation-theoretical studies (within the framework of more complex formulations) for finding approximate analytical solutions of various practically important problems connected with the investigation of the heat transfer at induced, free, and mixed convection.

NOTATION

C_p, heat capacity at a fixed pressure, J/(kg·K); C, constant; g, gravitational acceleration, m/sec^2; m, mass rate of flow, kg/(m·sec); Pr, Prandtl number; Q_0, excess heat content flow, J/(m·sec); T, temperature, K; u, v, longitudinal and transverse velocity components, m/sec; x, y, longitudinal and transverse coordinates, m; γ, temperature coefficient, 1/K^2; $\Delta T = T - T_\infty$, excess temperature, K; μ, dynamic viscosity coefficient, kg/(m·sec); ν, kinematic viscosity coefficient, m^2/sec; ρ, density, kg/m^3. Subscripts: ∞, surrounding liquid; m, maximum value; outer, outer region.

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